Enumerative Combinatorics –

In other words, Counting!

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Permutations and Combinations

Problem 1.1 How many ways can the letters of the word MATHS be arranged in a row?

There are:

- 5 choices for the first letter;
- Then, 4 choices remaining for the second letter;
- Then, 3 choices remaining for the third letter;
- Then, 2 choices remaining for the fourth letter;
- Then, only 1 choice remaining for the fifth letter.

In total, the number of arrangements (or *permutations*), is

$$5! \;=\; 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

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In general, the number of ways in which n different objects can be arranged in a row is

$$n! \;=\; n \times (n-1) \times (n-2) \cdots \times 3 \times 2 \times 1.$$

This notation is called "n factorial".

Problem 1.2 How many ways can the letters of the word HAPPY be arranged in a row?

Here the two letters P may be regarded as identical. If they were not identical, we would have 120 permutations as before. But each permutation is counted twice. Therefore the number of different permutations is

$$\frac{5!}{2!} = 60.$$

Problem 1.3 How many ways can the letters of the word HAPPPY be arranged in a row?

By a similar reasoning to the above, the answer is

$$\frac{6!}{3!} = 120.$$

If a set $A = \{a_1, a_2, \ldots, a_k\}$, then the size of A is denoted $|A|$ or $\#A$.

For example, if $A = \{2, 6, 7\}$, then $|A| = 3$.

Definition 1.4 For $n \geq 1$ and $0 \leq k \leq n$, the number of subsets of the set $\{1, 2, 3, \ldots, n\}$ of size k is

$$\binom{n}{k} := |\{A \subseteq \{1, 2, \ldots, n\} : A \text{ has } k \text{ elements}\}| .$$

Notice that choosing a subset is equivalent to choosing a team of k people from n people.

The numbers $\binom{n}{k}$ are called *binomial coefficients* and are central in enumerative combinatorics (counting). For completeness, we also define $\binom{0}{0} = 1$.

Example 1.5 For all natural numbers $n \geq 0$, $\binom{n}{0} = \binom{n}{n} = 1$.

It is easy to see that $\binom{n}{0} = 1$ for all numbers n. This is because there is only one subset of the set $\{1, 2, \ldots, n\}$ of size 0. It is the empty set $\emptyset$.

Similarly $\binom{n}{n} = 1$ for all natural numbers n because there is only one subset of $\{1, 2, \ldots, n\}$ of size n, namely the set $\{1, 2, \ldots, n\}$ itself.

Example 1.6 If we write down all of the 3-element subsets of $\{1, 2, 3, 4, 5\}$ we get

$$\begin{array}{c}\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \\ \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}.\end{array}$$

Thus we have $\binom{5}{3} = 10.$

How do we enumerate $\binom{n}{k}$ without constructing all the possible sets?

Let's go back to our example. Notice that choosing a subset is equivalent to choosing a team of 3 people from 5 people. We can arrange the 5 players in a row, and the three players who appear first in the row can be on the team. There are $5!$ arrangements for the row, but notice that every team will be counted $2! \times 3!$ times, since the three people on the left can be rearranged as we like (in $3!$ ways) without changing the team, and the last two people can also be rearranged as we like (in $2!$ ways) without changing the team.

We conclude that

$$\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10.$$

The same shows that in general:

$$\binom{n}{k} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{k \times (k-1) \times \cdots \times 2 \times 1} = \frac{n!}{k!\,(n-k)!}. \tag{1}$$

Problem 1.7 In how many ways can we choose a team of 3 people from 8 people?

The answer is

$$\binom{8}{3} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = \frac{8!}{3!\,5!} = 56.$$

We can arrange the numbers $\binom{n}{k}$ to form Pascal's triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
 & \vdots & & & & & & & & \vdots &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & \vdots & & & & & & \vdots & &
\end{array}
$$

Some properties of the binomial coefficients:

- For all natural numbers n and k with $n \geq 0$ and $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

  Specifying which k people we choose for the team is exactly the same as specifying the $n - k$ people we leave behind (i.e., who are *not* on the team).

  Problem 1.8 Find an alternative proof of this fact using Equation (1).

- For all natural numbers n, k where $0 \le k < n$,

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Suppose we have $n+1$ people, including Tom. We want to choose a team of $k+1$ people out of these. There are $\binom{n+1}{k+1}$ ways to do this. Now, let's count these teams in a different way.

If we include Tom, then we have n people left over and we need to choose k. There are $\binom{n}{k}$ ways to do this.

If we do *not* choose Tom, then we have n people left over and we need to choose $k+1$. There are $\binom{n}{k+1}$ ways to do this.

We conclude that

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Problem 1.9 Find an alternative proof of this fact using Equation (1).

Problem 1.10 How many ways can the letters of the word MISSISSIPPI be arranged in a row?

Problem 1.11 In a set there are n different blue objects and m different red objects. How many pairs of objects of the same colour can be made?

The Binomial Theorem:

Theorem 1.12 For all natural numbers n and real numbers x,

$$\sum_{i=0}^{n} \binom{n}{i} x^i \;=\; (1+x)^n.$$

The L.H.S. is 'sigma notation' for the expression

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

Example 1.13 For $n = 2$ we have

$$\binom{2}{0} + \binom{2}{1}x + \binom{2}{2}x^2 \;=\; (1+x)^2.$$

This is easily verified by expanding the R.H.S.

Example 1.14 Prove that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1} .$$

Example 1.15 $2n$ tennis players participate in a tournament, where $n \geq 1$. In how many ways can they paired up to play simultaneously?

Answer: $\dfrac{(2n)!}{2^n n!}$.

Example 1.16 Prove the following identities:

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1} \qquad (1 \le r \le n)\ .$$

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k} \qquad (1 \le k \le r \le n)\ .$$

$$\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1} \qquad (n \ge 1)\ .$$

Exercise 1.17 A spider has one sock and one shoe for each of its 8 legs. In how many different orders can the spider put on its socks and shoes, assuming that on each leg, the sock must be put on before the shoe?

Exercise 1.18 Prove the following identities:

$$\sum_{r=0}^{n} r^2 \binom{n}{r} = n(n+1)2^{n-2} \qquad (n \geq 2)\ .$$

$$\binom{n}{r} = \frac{n}{n-r}\binom{n-1}{r} \qquad (n \geq 2, 1 \leq r < n)\ .$$

Exercise 1.19 A number of boys and girls are seated at a round table. We know that there are 7 girls who have a girl to their right and 12 girls who have a boy to their right. We also know that 3 out of 4 of the boys have a girl to their right. How many children are seated at the table?

Exercise 1.20 A shop has red, green and blue hats.

(i) How many ways are there to choose 10 hats, assuming at least 10 hats of each type are available?

(ii) What if there are only 3 red, 4 green and 5 blue hats left in the shop?

Exercise 1.21 (BMO Round 1, 2005) Adrian teaches a class of six pairs of twins. He wishes to set up teams for a quiz, but wants to avoid putting any pair of twins into the same team. Subject to this condition:

(i) In how many ways can he split them into two teams of six?

(ii) In how many ways can he split them into three teams of four?

Exercise 1.22 (IMO 1981) Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each of these subsets has a smallest member. Let $F(n, r)$ denote the arithmetic mean (average) of these smallest numbers. Prove that

$$F(n, r) = \frac{n+1}{r+1}\,.$$

Exercise 1.23 (IMO 1987) Let n be a positive integer. Let $p_n(k)$ be the number of permutations of the set $\{1, 2, \ldots, n\}$ which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!$$

(Remark: A permutation f of a set S is a one-to-one mapping of S onto itself. An element i in S is called a fixed point of the permutation f if $f(i) = i$.)

Hints and Answers:

1. Exercise 1.17: Hint: Consider all orderings of the 16 items (socks and shoes), then think about how to get rid of the orderings that are "invalid" (i.e., where shoes are put on before socks). Answer: $\dfrac{(16)!}{2^8}$.
2. Exercise 1.18: Hint for first identity: consider committees of size r, where each committee has a Chairperson and a Secretary (which could possibly be the same person). Hint for second identity: first multiply both sides of the identity by $n - r$, then try to interpret what is being counted by each side.
3. Exercise 1.19: Answer: 35.
4. Exercise 1.20: Hint for part (i): Consider 12 items in a row. We can think of these as representing the 10 hats, together with two "dividers", such that the hats (if any) on the left of the first divider are red, the hats (if any) between the first and second divider are green, and the hats on the right of the second divider are blue. Answers: (i) 66; (ii) 6.

5. Exercise 1.21: Hint for first part: call the teams A and B. Now, for each pair of twins, one twin needs to be assigned to team A and the other to team B. Hint for second part: call the teams A, B and C. Consider how many ways there are to pick team A. Now, for each choice of team A, how many choices are left for team B? At the end, remember to get rid of the ordering on A, B and C. Answers: (i) 32; (ii) 960.
6. Exercise 1.22: Hint: for any r, how many subsets will have r as the smallest element? How is this useful in solving the problem?

Useful links:

1. Irish Maths Olympiad (IrMO) website: Click Here
2. IrMO past papers (excellent source of good problems): Click Here
3. Good textbook on problem solving, “Problem Solving Strategies” by Arthur Engel: Click Here
4. Good textbook on combinatorics, “Introduction to Combinatorics” by Leversha and Rowland: Click Here